The Time-Preference Nash Solution

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Abstract

We give an axiomatic characterization of the Time-Preference Nash Solution, a bargaining solution that is applied when the underlying preferences are defined over streams of physical outcomes. This bargaining solution is similar to the ordinal Nash solution introduced by Rubinstein, Safra, and Thomson (1992), but it gives a different prediction when the set of physical outcomes is a set of lotteries. JEL Classification, C72, C78.
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1 Introduction

In his seminal paper, Nash (1950) models a bargaining situation by concentrating on the set of available utility pairs and abstracting away from most of the details of the underlying environment. Thus the primitives of a bargaining problem consist of a set, S, of feasible utility pairs and a disagreement point in it. The idea is that the set S is induced by the lotteries over an underlying set of physical outcomes which, for the purposes of the analysis, can be abstracted away. Within this model Nash (1950) provides an axiomatic characterization of what is now the widely known Nash bargaining solution. Rubinstein, Safra, and Thomson (1992) (RST in the sequel) recast the bargaining problem into the underlying set of physical alternatives and give an axiomatization of what is known as the ordinal Nash bargaining solution. This solution has a very natural interpretation and has the interesting property that when risk preferences satisfy the expected utility axioms, it induces the standard Nash bargaining solution of the induced bargaining problem. This property justifies the proper name in the solution's appellation. Dealing with the question of existence and uniqueness, Hanany and Safra (2000) identify a large family of preferences within which the set of Nash outcomes is nonempty. Further, they show that if a preference relation is not in this family, then there is a preference relation in the family such that the corresponding bargaining problem has no Nash outcome. Burgos, Grant, and Kajii

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(2001) analyze an extensive form bargaining game whose subgame perfect equilibrium outcome converges to the ordinal Nash outcome as the risk of breakdown in negotiations tends to 0.

As Grant and Kajii (1995) (henceforth GK) note, although the characterizing properties used in RST are satisfied over a class that contains the expected utility preferences, they are not satisfied by a wide variety of interesting non-expected utility preferences. GK partially solve this problem by giving an alternative characterization of the ordinal Nash bargaining solution that holds for a class of preferences that contains the class covered by RST's assumptions. More significantly, GK show that every ordinal bargaining problem in that class, and not only those that are defined by expected utility preferences), induces a standard (cardinal) problem. Further, the ordinal Nash bargaining solution induces the standard Nash bargaining solution in the induced cardinal problems. As GK admit, the class of preferences for which their construction works is very restrictive. Further, one can argue that the GK construction does not yield the most natural set of utility pairs. To make the point clear, consider the following example, which is based on Example 1 in Grant and Kajii (1995).

Example 1 Two individuals bargain over one unit of a single commodity (money). Any non-negative division of the single commodity is feasible if both agents agree, otherwise they get 0. As a result, the set of physical outcomes is

$$X = \{(x_1, x_2) : x_1 + x_2 \leq 1,\ x_1, x_2 \geq 0\},$$

and the disagreement physical outcome is

$$D = (0, 0).$$

We shall assume that the bargainers are not constrained to agree on deterministic outcomes, but they can agree upon any lottery over elements of X. A *lottery*, ℓ, is a probability measure on the Borel sets of X. The set of lotteries will be denoted by $\mathcal{L}$. Assume that both bargainers' preferences over risky prospects satisfy the axioms of Yaari's (1987) dual theory of choice under risk. In particular, assume that agent i's risk preferences can be represented by the function

$$U_i(\ell) = \int_0^1 [G_\ell^i(t)]^{\alpha_i}\, dt \qquad i = 1, 2, \tag{1}$$

where for $i = 1, 2$, $\alpha_i > 1$ and G_ℓ^i is the decumulative distribution function of the i's payoffs.

Given $y \in X$ and $p \in [0, 1]$, we denote by $[y, p]$ the simple lottery that awards y with probability p and $(0, 0)$ otherwise. A Nash outcome (see Rubinstein, Safra, and Thomson (1992)) is an outcome $x^* \in X$ such that for all $p \in [0, 1]$ and for all $y \in X$,

$$U_i([p; y]) > U_i(x^*) \Rightarrow U_j([p; x^*]) > U_j(y) \qquad i \neq j, \quad i, j = 1, 2. \tag{2}$$

The interpretation of (2) is simple: if an agent is willing to run a risk of a breakdown in negotiations and get the outcome y with probability p instead of getting x^* with certainty, then the other agent is willing to run the same risk to get x^* with probability p instead of getting y with certainty. In our case condition 2 reduces to

$$p^{\alpha_i} y_i > x_i^* \Rightarrow p^{\alpha_j} x_j^* > y_j \qquad , i \neq j, \quad i, j = 1, 2.$$

As GK show, the only Nash outcome in this situation is given by

$$x^* = \arg\max\{x_1^{1/\alpha_1} x_2^{1/\alpha_2} : (x_1, x_2) \in X\}.$$

Let's now apply the standard Nash bargaining solution to our bargaining situation. For this purpose we need first to translate the bargaining situation into the utility space and define the corresponding bargaining problem $\langle S, d \rangle$. A natural choice of the utility possibilities set is given by the set of all utility pairs that can be achieved by means of a feasible lottery, namely

$$S = \{(U_1(\ell), U_2(\ell)) : \ell \in \mathcal{L}\}$$

and the corresponding disagreement point is $d = (0, 0)$. It is not difficult to see that in our case $S = \mathrm{co}\{(0, 0), (1, 0), (0, 1)\}$, and therefore the Nash bargaining solution picks $(\frac{1}{2}, \frac{1}{2})$, a utility pair that can be implemented only by the agreement $x = (\frac{1}{2}, \frac{1}{2}) \in X$. As we can see, RST and Nash's solutions give different outcomes.

GK argue that our set S is not the relevant set of utility pairs. With a very compelling argument they propose that the relevant utilities possibilities set is the following one:

$$S' = \{(s_1, s_2) : \exists (x_1, x_2) \in X,\ s_i \leq x_i^{1/\alpha_i},\ i = 1, 2\}.$$

Indeed, when we apply the Nash bargaining solution to $\langle S', 0 \rangle$, we get the utility pair that corresponds to RST's solution. Still, our set S seems to be very natural and the question is whether there is a solution concept defined over a class of ordinal problems, that induces the Nash bargaining solution over our set S.

In order to answer this question, we can find a hint in Binmore, Rubinstein, and Wolinsky (1986). There it is argued that there are two ways of constructing a cardinal bargaining problem out of a given set of physical outcomes, depending on whether risk- of time-preferences are involved. Once the appropriate bargaining problem is built, the solution that selects the point at which the Nash product is maximized is called the standard Nash solution or the time-preference Nash solution, respectively. Let's see now what happens if we add a time dimension to the problem in Example 1.

Example 1 (continued): Let $T = [0, \infty)$ be the time axis and assume that the agents have preferences over $\mathcal{L} \times T$ that can be represented by the following utility functions:

$$\delta^t U_i(\ell), \qquad i = 1, 2$$

where $\delta \in (0, 1)$. These preferences exhibit impatience in the sense that any given lottery is better sooner than later. Define ℓ^* to be a *time-preference Nash outcome* if for all $t \in T$ and for all $\ell \in \mathcal{L}$ we have

$$\delta^t U_i(\ell) > U_i(\ell^*) \Rightarrow \delta^t U_j(\ell^*) > U_j(\ell), \qquad i \neq j, \quad i, j = 1, 2. \tag{3}$$

The interpretation of (3) is simple. If an agent prefers to wait a period of length t in order to get a lottery ℓ rather than getting ℓ^* immediately, then the other agent prefers to insist on ℓ^* for the same lapse of time rather than agreeing to ℓ immediately. As will be shown in a later section, the only time-preference Nash outcome in our bargaining situation is the certain division $x = (\frac{1}{2}, \frac{1}{2}) \in X$, the same one obtained within the cardinal approach when applied to the set S.

The purpose of this paper is to give an axiomatic characterization of the rule that assigns the time-preference Nash outcome to each bargaining problem. The idea is analogous to the one applied by RST. The primitives of RST's model is an abstract set X of physical alternatives, and a status-quo outcome $D \in X$. RST's aim is to define a solution concept F that maps each pair of preference relations to a unique outcome in X which has some relation to the cardinal Nash bargaining solution. The problem is that the preference relations over the set X are not enough to perform this task. As a result, RST consider extensions of the preferences over X to the set of *lotteries* over X. In other words, they enlarge the model to allow for lotteries whose outcomes are elements of X, and in this way they are able to define a solution concept F that, while it is defined over pairs of risk preferences, still selects an element of X, namely a non-degenerate lottery. Our strategy differs from RST's in that we do not extend the agents' preferences over X to risk preferences but to *time preferences*. In other words, our solution concept will take a pair of preferences over time-combinations of elements of X and still return an element of X, namely an immediate agreement.

There are some advantages in our characterization of the time-preference Nash solution. The main advantage is that when the basic set of physical alternatives is a set of lotteries, no special assumptions need to be made about the agents' risk preferences. Therefore all of the widely used non-expected utility preferences are, in principle, covered.

In principle, an axiomatization of the time-preference Nash solution could be done by performing a mechanic adaptation of the axioms used in RST or in GK. But our characterization uses, along with some standard axioms, axioms that are related to time. One axiom requires that if one agent becomes more impatient in some well-defined sense, and the other becomes less impatient, then the solution should change in favor of the agent that became more patient. The other axiom has a flavor of subgame perfection. At any particular but arbitrary point in time t, the players have the choice of disagreeing for ever, but still the solution picks an agreement x. Consider a situation where if there is no agreement by time t to accept the solution's recommendation. It is as if the disagreement point, instead of being disagreement for ever, consisted of agreement x at time t. The axiom requires from a solution to chose the same outcome x both if the status quo is disagreement for ever or if it is agreeing to x at time t.

As should be clear from Example 1, RST's ordinal Nash solution and the time-preference Nash solutions, when applied to a set of lotteries, give different predictions. In fact, as Volij and Winter (1999) show, they even have very different comparative statics properties (see also Safra and Zilcha (1993) for related results). In particular, risk aversion plays no role in our example when the time-preference Nash solution is applied while risk aversion has an influence when RST's ordinal Nash solution is applied.

2 The Model

Let X be a set of physical alternatives and let $\underline{d}$ be an element of X. The set X is supposed to be a non empty compact subset of a vector space and $\underline{d}$ represents the status-quo outcome. Let $T = [0, \infty)$ be the time axis. An *agreement* is a function $x : T \to X$ that determines the physical outcome enforced at every point in time. An *elementary agreement* is a constant agreement, namely, a function x such that $x(t) = \bar{x}$ for some $\bar{x} \in X$. We shall identify the set of elementary agreements with X. Let x and y be two agreements and let τ be a point in time. We denote by $[y\tau x]$ the agreement defined by

$$y\tau x(t) = \begin{cases} y(t) & \text{if } t < \tau \\ x(t-\tau) & \text{if } t \geq \tau. \end{cases}$$

That is, $[y\tau x]$ postpones x until τ and coincides with y for 0 until τ. We say that $[y\tau x]$ is a *time combination* of x and y. Although an agreement is any evolution of physical outcomes over time, for simplicity we shall restrict attention to agreements of the form $[x, \tau y]$ where both x and y are elementary agreements. Therefore, a feasible agreement $[y\tau x]$ can be interpreted as outcome y until time τ, and outcome x from τ on. Denote by $\mathcal{X}$ the set of feasible agreements.

Let $\delta \in (0, 1)$. In this paper we shall restrict attention to preferences over $\mathcal{X}$ that, for some continuous function $u : X \to \mathbb{R}$ with $u(x) \geq u(\underline{d})$ for all $x \in \mathcal{X}$, can be represented by the following functional form:

$$U(x) = \delta \int_0^\infty e^{-\delta t} u(x(t))\, dt, \tag{4}$$

which, given our restriction to $\mathcal{X}$, can be written as

$$U(y\tau z) = (1 - e^{-delta\tau})u(y) + e^{-\delta\tau}u(z).$$

The number δ, which will be fixed throughout the analysis, is the agents' common discount factor. The requrement that $u(x) \geq u(\underline{d})$ means that the worst posible outcome is $\underline{d}$ forever. Note that $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ represent the same preferences over $\mathcal{X}$, if and only if $v = au + b$ for all $a > 0$ and for all b. Denote by $\mathcal{P}$ the set of continuous real functions on X such that $u(x) \geq u(\underline{d})$, for all $x \in X$. Let u_1 and u_2 be a pair of functions in $\mathcal{P}$. We say that an outcome $x \in X$ is *efficient* if there is no outcome $y \in X$ with $u_i(y) > u_i(x)$ for both $i = 1, 2$.

Definition 1 A *negotiation problem* is a pair $\langle (u_1, u_2), D\rangle$ where (u_1, u_2) is a pair of utility functions in $\mathcal{P}$ and $D \in X$ is a status-quo agreement, which is not efficient.

We use the expression negotiation problem to denote a pair $\langle (u_1, u_2), D\rangle$ because we want to reserve *bargaining problem* to denote the problems analized by Nash (1950). The interpretation of a negotiation problem is as follows: u_i represents the time-preference of agent i, and D is the outcome that represents disagreement for ever. We shall sometimes denote a negotiation problem $B = \langle (u_1, u_2), D\rangle$ by $\langle u, D\rangle$, where $u : X \to \mathbb{R}^2$ is defined by $u(x) = (u_1(x), u_2(x))$. Further, for a subset $Y \subseteq X$, $u(Y) = \{s \in \mathbb{R}^2 : \exists x \in Y \text{ such that } s = u(x)\}$ is the image of Y under u.

Let $B = \langle u, D\rangle$ be a negotiation problem. An outcome $x \in X$ is *individually rational* if $u(x) \geq u(D)$. It is *strongly individually rational* if $u(x) > u(D)$. We denote by $\mathcal{IR}(B)$ the set of individually rational agreements.

Definition 2 Let $B = \langle (u_1, u_2), D\rangle$ be a negotiation problem. A Nash agreement is an individually rational outcome $x^* \in \mathcal{IR}(B)$ such that there is no agent i, for $i = 1, 2$, time τ and outcome $y \in \mathcal{IR}(B)$ such that $U_i([D\tau y]) > u_i(x^*)$ and $u_j(y) > U_j([D\tau x^*])$.

A Nash agreement is the time-preference version of RST's ordinal Nash outcome. The interpretation is simple: if one agent would rather wait for τ time-units to get y than agreeing to x^* immediately, then the other agent would rather wait for the same lapse of time to get x^* than agreeing on y immediately.

The following lemma shows that a Nash agreement is efficient and strongly individually rational and it will be useful in the Nash agreement's cardinal characterization.

Lemma 1 Let $B = \langle u, D\rangle$ be a negotiation problem and let x^* be a Nash agreement of B. Then x^* is efficient and strongly individually rational.

Proof : It follows directly from the definition that a Nash agreement cannot be inefficient. Assume now that for some i, $u_i(x^*) = u_i(D)$. Since D is not efficient, there is an outcome $y \in X$ such that $u_j(y) > u_j(D)$ for $j = 1, 2$. Therefore, for all τ we have $U_i([D\tau y]) > u_i(x^*)$. On the other hand, since $\lim_{t\to\infty} U_j([Dtx^*]) = u_j(D)$, there must be a τ big enough such that $u_j(y) > U_j([D\tau x^*])$. Consequently x^* cannot be a Nash agreement. □

We shall restrict attention to negotiation problems that satisfy the following properties:

Convexity: A negotiation problem $\langle u, D\rangle$ is *convex* if for all $x \in \mathcal{X}$ there is an elementary agreement $z \in X$ such that $u_i(z) = U_i(x)$ for $i = 1, 2$.

Parsimony: A negotiation problem $\langle u, D\rangle$ is *parsimonious* if $x = y$ whenever $u(x) = u(y)$.

Free disposal: A negotiation problem $B = \langle (u_1, u_2), D\rangle$ satisfies *free disposal* if for all $x \in \mathcal{IR}(B)$, there are z_1 and z_2 in $\mathcal{IR}(B)$ such that $u_i(z_i) = u_i(x)$ and $u_i(z_j) = u_i(\underline{d})$ for $i = 1, 2$ and $j \neq i$.

Convexity says that for every agreement there is a constant immediate agreement that is utility equivalent. It is not an innocuous assumption that allows us to restrict attention to constant immediate agreements. Parsimony allows us to get rid of irrelevant multiplicity and free disposal will allow us to deal with a comprehensive feasible set. Parsimony and Free disposal are mainly simplifying assumptions. We believe that they are not crucial for our results.

Denote by $\mathcal{N}$ the set of all convex, parsimonious negotiation problems that satisfy free disposal.

The following useful result is standard.

Lemma 2 Let $B = \langle (u_1, u_2), D\rangle$ be a negotiation problem. The elementary agreement x^* is a Nash agreement if and only if it solves

$$\max_{x \in \mathcal{IR}(B)} (u_1(x) - u_1(D))(u_2(x) - u_2(D)). \tag{5}$$

Proof : Let $x^* \in \mathcal{IR}(B)$ and assume there exists an agent i, outcome $y \in \mathcal{IR}(B)$ and time τ such that

$$U_i([D\tau y]) > u_i(x^*) \quad \text{and} \quad u_j(y) > U_j([D\tau x^*]). \tag{6}$$

This means

$$(1 - e^{-\delta\tau})u_i(D) + e^{-\delta\tau}u_i(y) > u_i(x^*) \text{ and } u_j(y) > (1 - e^{-\delta\tau})u_i(D) + e^{-\delta\tau}u_i(x^*).$$

Rearranging,

$$e^{-\delta\tau}(u_i(y) - u_i(D)) > u_i(x^*) - u_i(D) \text{ and } u_j(y) - u_j(D) > e^{-\delta\tau}(u_j(x^*) - u_j(D)).$$

Multiplying both inequalities,

$$e^{-\delta\tau}(u_i(y) - u_i(D))(u_j(y) - u_j(D)) > e^{-\delta\tau}(u_j(x^*) - u_j(D))(u_i(x^*) - u_i(D))$$

which means that x^* is not a solution to (5).

Conversely, assume that x^* is not a solution to (5). If x^* is not strongly individually rational, then by Lemma 1 x^* is not a Nash outcome. If x^* is strongly individually rational, then $u_i(x^*) > u_i(D)$ for $i = 1, 2$. Therefore we can find a time τ and outcome $y \in \mathcal{IR}(B)$ such that

$$\frac{u_i(y) - u_i(D)}{u_i(x^*) - u_i(D)} > e^{-\delta\tau} > \frac{u_j(x^*) - u_j(D)}{u_j(y) - u_j(D)}.$$

Then

$$u_i(y) - u_i(D) > e^{-\delta\tau}(u_i(x^*) - u_i(D))$$

and

$$e^{-\delta\tau}(u_j(y) - u_j(D)) > u_j(x^*) - u_j(D).$$

This means

$$\begin{aligned} u_i(y) &> (1 - e^{-\delta\tau})u_i(D) + e^{-\delta\tau}u_i(x^*) \\ u_j(x^*) &< (1 - e^{-\delta\tau})u_j(D) + e^{-\delta\tau}u_j(y). \end{aligned}$$

Namely x^* is not a Nash outcome. □

As a corollary of the previous Lemma 2 we get the following:

Proposition 1 Let $B = \langle u, D \rangle$ be a negotiation problem in $\mathcal{B}$. Then a Nash agreement exists. Further, if B is convex and parsimonious, the Nash agreement is unique.

Proof : Assume without loss of generality that $u_1(D) = u_2(D) = 0$. Since u_1 and u_2 are continuous functions and X is a compact set, the problem defined in (5) has a solution. By Lemma 2 there is a Nash outcome x. Assume that y is another Nash outcome. Let $\tau > 0$ and consider the simple agreement $[y\tau x]$. Since the problem is convex, there is an elementary agreement $z \in X$ such that $u_i(z) = U_i([y\tau x])$, for $i = 1, 2$. Then we must have

$$\begin{aligned} u_1(z)u_2(z) &\geq ((1 - e^{-\delta\tau})u_1(y) + e^{-\delta\tau}u_1(x))((1 - e^{-\delta\tau})u_2(y) + e^{-\delta\tau}u_2(x)) \\ &\geq u_1(x)u_2(x) = u_1(y)u_2(y) \end{aligned}$$

where the inequlaity follows from the strict quasi-concavity of the function $f(v_1, v_2) = v_1 v_2$. Further, the inequality is strict unless x and y are utility equivalent. But since x and y are Nash outcomes, we must have equality, which implies that they are utility equivalent. But since B is parsimonious, $x = y$. $\square$

A *negotiation solution* is a function $F : \mathcal{N} \to X$ that assigns an outcome to each negotiation problem.

Given Proposition 1, the following is a well-defined solution.

Definition 3 The *time-preference Nash solution* is the solution that assigns to each negotiation problem in $\mathcal{N}$, its time-preference Nash outcome.

3 Characterization

3.1 The axioms

We now turn to the axiomatic characterization of the time-preference Nash solution. We are interested in appealing properties that are also adequate for our framework of time-contingent agreements. We start with the two standard axioms of efficiency and symmetry.

EFF: A negotiation solution F satisfies *efficiency* if for all $B \in \mathcal{N}$, $F(B)$ is efficient.

A negotiation problem $\langle (u_1, u_2), D \rangle$ is *symmetric* if there exists a bijection $\phi : X \to X$ such that

1. $\phi^{-1} = \phi$
2. $\phi(D) = D$
3. for all $x, y \in X$ and for all $\tau, \tau' \in T$, $U_1([D\tau x]) > U_1([D\tau' y]) \Leftrightarrow U_2([D\tau\phi(x)]) > U_2([D\tau'\phi(y)])$.

SYM: A negotiation solution F satisfies *symmetry* if for all symmetric problems B with symmetry function ϕ, $F(B) = \phi(F(B))$.

Claim 1 The time-preference Nash solution satisfies symmetry.

Proof: Let $B = \langle (u_1, u_2), D \rangle \in \mathcal{N}$ be a symmetric negotiation problem with symmetry function ϕ and let x^* be its time-preference Nash outcome. We must show that $\phi(x^*) = x^*$. First note that $\phi(x^*) \in \mathcal{IR}(B)$. Indeed, since $x^* \in \mathcal{IR}(B)$, we have that

$$u_1(x^*) \geq u_1(D) \quad \text{and} \quad u_2(x^*) \geq u_2(D)$$

which, since ϕ is a symmetry function, implies that

$$u_2(\phi(x^*)) \geq u_2(D) \quad \text{and} \quad u_1(\phi(x^*)) \geq u_1(D).$$

Assume by contradiction that $\phi(x^*)$ is not a time-preference Nash outcome. Then there is $y \in \mathcal{IR}(B)$, $i \in \{1,2\}$ and $\tau \in T$ such that

$$U_i([D\tau y]) > u_i(\phi(x^*)) \quad \text{and} \quad u_j(y) > U_i([D\tau\phi(x^*)]).$$

But since ϕ is a symmetry function we get

$$U_j([D\tau\phi(y)]) > u_i(x^*) \quad \text{and} \quad u_j(\phi(y)) > U_i([D\tau x^*])$$

which contradicts the fact that x^* is a time-preference Nash outcome. $\square$

We know that two functions u and u' represent the same preferences over $\mathcal{X}$ if one can be obtained from the other by means of a positive affine transformation. It would be reasonable then to require that the solution be invariant to equivalent utility representations. In order to state this requirement, we need the following definition.

Two negotiation problems $B = \langle(u_1, u_2), D\rangle$ and $B' = \langle(u'_1, u'_2), D\rangle$ are *equivalent* if there are $\alpha_i > 0$ and β_i, for $i = 1, 2$ such that $u'_i = \alpha_i u_i + \beta_i$, for $i = 1, 2$.

INV: A negotiation solution F satisfies *invariance* if for all equivalent problems $B = \langle(u_1, u_2), D\rangle$ and $B' = \langle(u'_1, u'_2), D\rangle$, $F(B) = F(B')$.

We now present other axioms, some of which make use of the time-preference nature of the problem.

INIRA: A negotiation solution F satisfies *independence of non-individually rational outcomes* if for all pair of problems $B = \langle u, D\rangle, B' = \langle u', D\rangle$ in $\mathcal{N}$ such that

- $\mathcal{IR}(B) = \mathcal{IR}(B')$
- for all $x, \in \mathcal{IR}(B)$, $u(x) = u'(x)$

we have $F(B) = F(B')$.

This axiom requires that the solution be dependent on the agents' preference relations over the individually rational outcomes only. It is easy to check the validity of the following:

Claim 2 The time-preference Nash solution satisfies INV and INIRA.

Consider the negotiation problem $\langle u, D \rangle$ and assume the solution recommends the agreement x^*. D here represents the outcome of disagreement forever. Suppose that the agents agree that if they do not reach an agreement before time t, then the outcome will be x^* from t on. In this case, we can regard $[Dtx^*]$ as the status-quo. The next axiom requires from a solution to recommend the same outcome both when we consider D or $[Dtx^*]$ as the disagreement outcome.

TFP: A negotiation solution F has the *tree-folding property* if for all negotiation problems $\langle u, D \rangle$, we have that $F(u, D) = F(u, z)$ for all $z \in X$ such that for some $t \in T$, $u_i(z) = U_i([DtF(u, D)])$.

Suppose the $F(u, D)$ is the agreement proposed by the solution, then the axiom requires that the same agreement be selected if the disagreement outcome D is replaced by a postponed implementation of the agreement $F(u, D)$. We call this axiom the tree folding property because it is related the corresponding property of the Nash equilibrium concept of extensive form games. Consider an extensive form game and fix a Nash equilibrium σ in it. For every node n in the tree, σ determines an outcome, $z(n, \sigma)$, which is the outcome that would result if σ was played in the subgame that starts at node n. In particular, σ determines a Nash equilibrium outcome $z(n_0, \sigma)$, where n_0 denotes the root of the tree. Now, $z(n_0, \sigma)$ remains a Nash equilibrium outcome if we replace any given node n by the terminal history $z(n, \sigma)$. Needless to say, this "tree-folding property" is also satisfied by the Subgame Perfect equilibrium concept, but we want to stress that it is so basic that it is even satisfied by the Nash equilibrium concept. The outcome $F(u, D)$ in the TFP axiom, represents the subgame perfect equilibrium outcome of some extensive form, stationary bargaining game. The outcome $DtF(u, D)$, on the other hand, represents the outcome induced by the subgame perfect equilibrium for the subgames that start at time t. We know that if we replaced each subgame that starts at t by the subgame perfect equilibrium outcome $DtF(u, D)$, then the subgame perfect equilibrium outcome of the amended game would remain $F(u, D)$. This is exactly what the TFP axiom wants to capture.

Claim 3 The time-preference Nash solution satisfies the tree-folding property.

Proof : Let $B = \langle (u_1, u_2), D \rangle \in \mathcal{N}$ be a negotiation problem and let x^* be its time-preference Nash agreement. Then, by Lemma 2, x^* solves

$$\max_{y \in \mathcal{IR}(B)} (u_1(y) - u_1(D))(u_2(y) - u_2(D)).$$

Let $z \in X$ and $\tau \in T$ be such that $u_i(z) = (1 - e^{\delta\tau})u_i(D) + e^{-\delta\tau}u_i(x^*)$ for $= 1, 2$ and let $B' = \langle (u_1, u_2), z \rangle$. By Peters and van Damme (1991) (Lemma 3.2), x^* solves

$$\max_{y \in \mathcal{IR}(B')} (u_1(y) - u_1(z))(u_2(y) - u_2(z)).$$

Therefore, by Lemma 2, x^* is the time-preference Nash agreement of B'. $\square$

The next axiom compares two problems, B and B', with the same disagreement outcome. The difference between the two problems is that when we go from B to B' one player becomes more impatient and the second becomes less impatient. Denote by x and x' the outcomes recommended by the a solution to B and B', respectively. The axiom requires from the solution that x' be preferred to x by the agent who became more patient and that x be preferred to x' by the agent who became more impatient.

IM: A negotiation solution F satisfies *impatience monotonicity* if the following condition holds: whenever $B = \langle u, D \rangle$ and $B' = \langle u', D \rangle$ are two negotiation problems such that

1. $u(D) = u'(D)$ and $u(F(B)) = u'(F(B))$.
2. for all $x \in X$, if x is efficient and individually rational in B so is x in B'.
3. for all efficient and individually rational points $x, y \in X$, $u_i(x) \geq u_i(y)$ if and only if $u'_i(x) \geq u'_i(y)$, for $i = 1, 2$.
4. for some $i = 1, 2$ and $j = 3 - i$
   - if $u_i(F(B)) = U_i([D\tau_i x])$ then $u'_i(F(B)) \geq U'_i([D\tau_i x])$
   - if $u_j(x) = U_j([D\tau_j F(B)])$ then $U'_j([D\tau_j F(B)]) \geq u'_j(x)$.

then $u'_i(F(B')) \leq u'_i(F(B))$.

Claim 4 The time-preference Nash solution satisfies impatience monotonicity.

Proof : Let $B = \langle (u_1, u_2), D \rangle$ and $B' = \langle (u'_1, u'_2), D \rangle$ be two negotiation problems that satisfy the conditions of the axiom and let x^* be the time-preference Nash outcome of B. Let $x \in X$ be an efficient outcome such that $u'_i(x) > u'_i(x^*)$. We shall show that x is not a time-preference Nash outcome of B'. By condition 3, $u_i(x) > u_i(x^*)$, which by efficiency of the time-preference Nash solution implies that $u_j(x) < u_j(x^*)$. Again, by condition 3, $u'_j(x) < u'_j(x^*)$. Now, since $u_i(x) > u_i(x^*)$, there is $\tau \in T$ such that

$$(1 - e^{-\delta\tau})u_i(D) + e^{-\delta\tau}u_i(x) = u_i(x^*).$$

By 4 and 1,

$$(1 - e^{-\delta\tau})u'_i(D) + e^{-\delta\tau}u'_i(x) \leq u_i(x^*).$$

Therefore, it follows from the above two inequalities that

$$u'_i(x) - u'_i(D) \leq u_i(x) - u_i(D).$$

Similarly, since $u'_j(x) < u'_j(x^*)$, we have that there is $\tau' \in T$ such that

$$(1 - e^{-\delta\tau'})u_j(D) + e^{-\delta\tau'}u_j(x^*) = u_j(x).$$

By 4 and 1

$$(1 - e^{-\delta\tau'})u'_j(D) + e^{-\delta\tau'}u'_j(x^*) \geq u'_j(x).$$

Therefore, it follows from the above two inequalities and 1 that

$$u'_j(x) - u'_j(D) \leq u_j(x) - u'_j(D).$$

Then we have that

$$\begin{aligned}
(u'_1(x) - u'_1(D))(u'_2(x) - u'_2(D)) &\leq (u_1(x) - u_1(D))(u_2(x) - u_2(D)) \\
&< (u_1(x^*) - u_1(D))(u_2(x^*) - u_2(D)) \\
&= (u'_1(x^*) - u'_1(D))(u'_2(x^*) - u'_2(D)).
\end{aligned}$$

Therefore, by Lemma 2, x is not a time-preference Nash outcome of B'. □

We are now ready to state our main result.

Theorem 1 A negotiation solution $F : \mathcal{N} \to X$ satisfies efficiency, symmetry, invariance, independence of non-individually rational alternatives, the tree-folding property and impatience monotonicity, if and only if F is the time-preference Nash solution.

3.2 Proof of Theorem 1

The idea of the proof is similar to the one applied by Grant and Kajii (1995). We first define the family of bargaining problems (in the sense of Nash (1950)) induced by the family of negotiation problems we are dealing with. Next, for each negotiation solution we will define its associated bargaining solution: a set valued function that maps bargaining solutions into a subset of its feasible utilities set. Finally, we will show that if the negotiation solution satisfies the axioms of the theorem, then the associated bargaining solution will satisfy some corresponding properties which characterize the Nash bargaining solution.

The following definitions are standard. A *bargaining problem* is a pair (S, d) where $S \subseteq \mathbb{R}^2$ is a compact, convex set, $d \in S$ and there is $s \in S$ with[1] $s \gg d$. We denote by $\mathcal{B}$ the set of all bargaining problems. Let (S, d) be a bargaining problem. We say that $s \in S$ is *individually rational* if $s \geq d$. We say that $s \in S$ is *weakly efficient* if there is no $s' \in S$ such that $s' \gg s$. We denote by $\mathcal{IR}(S, d)$ the set of individually rational points in (S, d). A bargaining problem (S, d) is *d-comprehensive* if $s \geq s' \geq d$ and $s \in S$, then $s' \in S$. A bargaining problem (S, d) is *symmetric* if

- $d_1 = d_2$ and
- $(s_1, s_2) \in S$ implies $(s_2, s_1) \in S$.

We say that (S', d') is obtained from the bargaining problem (S, d) by the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1, 2$, if $d'_i = \alpha_i d_i + \beta_i$, for $i = 1, 2$ and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Definition 4 Let $B = \langle u, D \rangle$ be a negotiation problem in $\mathcal{N}$. The bargaining problem induced by B is the bargaining problem $\langle S, d \rangle$ where

- $d = u(D)$,
- $S = \{s \in \mathbb{R} : \exists x \in \mathcal{X} \text{ s.t. } s = U((x))\}$.

Note that since B is a convex negotiation problem we can state the above conditions as

- $d = u(D)$
- $S = u(X)$

Lemma 3 Let $B \in \mathcal{N}$ be a negotiation problem, its induced bargaining problem $\langle S, d \rangle$ is a comprehensive bargaining problem with a point $d_0 \in S$ such that $d_0 \leq s$ for all $s \in S$.

Proof : Since X is a compact set and u is continuous, $S(= u(X))$, as the continuous image of a compact set is compact. Since $D \in X$, $d = u(D) \in u(X) = S$. Since there is $x \in X$ such that $u(x) \gg u(D)$, we have that there is $s \in S$ such that $s \gg d$. Since $u(\underline{d}) \leq u(x)$ for all $x \in X$, $d_0 \equiv u(\underline{d}) \leq s$ for all $s \in S$. To show that S is convex let $s, s' \in S$. Then there exist $x, x' \in X$ such that

$$s_i = U_i(x) \quad \text{and} \quad s'_i = U_i(x').$$

[1] We adopt the following conventions for vector inequalities: $x \gg y \leftrightarrow x_i > y_i$ for all i, and $x \geq y \leftrightarrow x_i \geq y_i$ for all i.

Let $\alpha \in (0,1)$. There is a time $t \in T$ such that

$$\alpha s_i + (1-\alpha)s'_i = U_i([xtx']).$$

By convexity of B we have that there exists $z \in X$ such that $z \sim [xtx']$ which implies that $\alpha s + (1-\alpha)s' \in S$. It remains to show that $\langle S, d\rangle$ is comprehensive. But this follows immediately from the assumptions of free disposal and convexity. $\square$

Denote by $\mathcal{S}$ the set of all the cardinal problems induced by some convex negotiation problem $B \in \mathcal{N}$.

Lemma 4 The class $\mathcal{S}$ is the class of all cardinal bargaining problems $\langle S, d\rangle$ that are comprehensive, and with a point $d_0 \in S$ with $d_0 \leq s$ for all $s \in S$.

Proof : Given Lemma 3, it is enough to show that for every comprehensive bargaining problem $\langle S, d\rangle$, with a point $d_0 \in S$ such that $d_0 \leq s$ for all $s \in S$, there is a negotiation problem in $\mathcal{N}$ that induces $\langle S, d\rangle$. So let $B' = \langle (u'_1, u'_2), D\rangle$ be a negotiation problem in $\mathcal{N}$, and let $\langle S', d'\rangle$ be its induced cardinal problem. Without loss of generality we can assume that $d' = d$. For each $x \in X \setminus \{D\}$ define

$$\begin{aligned}
\lambda'(x) &= \max\{\lambda \geq 0 : (1-\lambda)u'(D) + \lambda u'(x) \in S'\} \\
\lambda(x) &= \max\{\lambda \geq 0 : (1-\lambda)u'(D) + \lambda u'(x) \in S\}.
\end{aligned}$$

Since S' and S are compact sets, the above numbers are well-defined. Further, since d is an interior point of S', we have $\lambda'(x) > 0$. We can therefore define $\alpha(x) = \lambda(x)/\lambda'(x)$ and the following functions on X:

$$u_i(x) = \begin{cases} u'_i(x) & \text{if } x = D \\ \alpha(x)u'_i(x) & \text{otherwise} \end{cases} \qquad i = 1, 2.$$

It can be checked that u is continuous and that $B = \langle (u_1, u_2), D\rangle$ is a negotiation problem that induces S. $\square$

Let F be a negotiation solution defined on $\mathcal{N}$ that satisfies efficiency, symmetry, invariance, independence of non-individually rational outcomes, impatience monotonicity, and the tree-folding property. Define the correspondence $f : \mathcal{S} \to 2^{\mathbb{R}^2} \setminus \emptyset$ by

$$f(S, d) = \{(u_1(F(B)), u_2(F(B))) : \exists B = \langle (u_1, u_2), D\rangle \in \mathcal{N} \text{ that induces } \langle S, d\rangle\}.$$

Lemma 5 The correspondence f is the Nash bargaining solution. Namely, f is the singleton $\{(s_1^*, s_2^*)\} \subseteq S$ such that $(s_1^* - d_1)(s_2^* - d_2) \geq (s_1 - d_1)(s_2 - d_2)$ for all $(s_1, s_2) \in S$ such that $s_i \geq d_i$ for $i = 1, 2$.

Proof : It is enough to show that f satisfies all the properties that, according to the main theorem in Dagan, Volij, and Winter (1999), characterize the Nash bargaining solution. These properties are the following:

Invariance: Whenever (S', d') is obtained from the bargaining problem (S, d) by means of the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1, 2$, where $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$, we have that $f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$, for $i = 1, 2$.

Weak Pareto optimality: For all bargaining problems (S, d), $f(S, d)$ is a subset of the weakly efficient points in S.

Symmetry: For all symmetric bargaining problems (S, d),

$$(s_1, s_2) \in f(S, d) \Leftrightarrow (s_2, s_1) \in f(S, d).$$

Single-valuedness in symmetric problems: For every symmetric problem $B \in \mathcal{B}$, $f(B)$ is a singleton.

Twisting: Let (S, d) be a bargaining problem and let $(\hat{s}_1, \hat{s}_2) \in f(S, d)$. Let (S', d) be another bargaining problem such that for some agent $i = 1, 2$

$$\begin{aligned} S \setminus S' &\subseteq \{(s_1, s_2) : s_i > \hat{s}_i\} \\ S' \setminus S &\subseteq \{(s_1, s_2) : s_i < \hat{s}_i\}. \end{aligned}$$

Then, there is $(s'_1, s'_2) \in f(S', d)$ such that $s'_i \leq \hat{s}_i$.

Disagreement point convexity: For every bargaining problem $B = (S, d)$, for all $s \in f(S, d)$ and for every $\lambda \in (0, 1)$ we have $s \in f(S, (1 - \lambda)d + \lambda s)$.

Independence of non-individually rational alternatives: For every two problems (S, d) and (S', d) such that $\mathcal{IR}(S, d) = \mathcal{IR}(S', d)$ we have $f(S, d) = f(S', d)$.

The first four properties are standard. The axioms of twisting and disagreement point convexity were introduced, in its single-valued versions, by Thomson and Myerson (1980) and Peters and van Damme (1991), respectively. Independence of non-individually rational alternatives is first discussed in Peters (1986). Now we turn to show that our correspondence f satisfies the above properties.

Invariance: Let $\langle S, d\rangle$ be a bargaining problem in $\mathcal{S}$ and let $s^* \in f(S, d)$. Then there is a negotiation problem $B = \langle (u_1, u_2), D\rangle \in \mathcal{N}$ and an outcome $x^* \in X$ such that $\langle S, d\rangle$ is induced by B and $s^* = u(F(B))$. Let $\langle S', d'\rangle$ be a cardinal problem that is obtained from $\langle S, d\rangle$ by means of the transformations $s_i \to \alpha_i s_i + \beta_i$, for $i = 1, 2$ and $\alpha_i > 0$. Then, the negotiation problem $B' = \langle (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2), D\rangle$ induces $\langle S', d\rangle$ and since F satisfies invariance, still $x^* = F(B')$. Therefore $(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in f(S', d')$.
Weak Pareto optimality: Let $(s_1, s_2) \in f(S)$. Then there is a negotiation problem B that induces S such that $s_i = u_i(F(B))$ for $i = 1, 2$. If there was $(s'_1, s'_2) \in S$ such that $(s'_1, s'_2) \gg (s_1, s_2)$, then there would be $y \in X$ such that $s'_i = u_i(y)$, for $i = 1, 2$ and $u_i(y) > u_i(F(B))$ for $i = 1, 2$. This would contradict the fact that F is efficient.
Symmetry and single-valuedness in symmetric problems: Let $\langle S, d\rangle$ be a symmetric cardinal problem. Then there is a negotiation problem $B = \langle u, D\rangle$ that induces it, namely $S = u(X)$ and $d = u(D)$. Let $x \in X$. Then $(s_1, s_2) = (u_1(x), u_2(x)) \in S$. Since S is symmetric there is $y \in X$ such that $(s_2, s_1) = (u_1(y), u_2(y)) \in S$ and, since B is assumed to be parsimonious, this y is unique. Define $\phi(x) = y$. We claim that ϕ is a symmetry function. To see this, note that by definition of ϕ, $\phi(\phi(x)) = x$ and $\phi(D) = D$. Further,

$$\begin{aligned}
U_1([D\tau z]) > U_1([D\tau' z']) \quad &\Leftrightarrow \quad (1 - e^{-\delta\tau})u_1(D) + e^{-\delta\tau}u_1(x) > (1 - e^{-\delta\tau'})u_1(D) + e^{-\delta\tau'}u_1(z') \\
&\Leftrightarrow \quad (1 - e^{-\delta\tau}))u_2(D) + e^{-\delta\tau}u_2(\phi(x)) > (1 - e^{-\delta\tau'}u_2(D) + e^{-\delta\tau'}u_2(\phi(z')) \\
&\Leftrightarrow \quad U_2([D\tau\phi(z)]) > U_2([D\tau'\phi(z')]).
\end{aligned}$$

This means that B is a symmetric negotiation problem. By symmetry of F, $\phi(F(B)) = F(B)$. We also have $u_2(x) = u_1(\phi(x))$ for all $x \in X$. Therefore $u_2(F(B)) = u_1(\phi(F(B)) = u_1(F(B))$, namely, the solution is on the 45 degree line. We can conclude then that f satisfies symmetry. Further, since F is efficient, $u(F(B))$ is the unique efficient point in the 45 degree line which means that f is single-valued for symmetric problems.
Disagreement point convexity: Let $\langle S, d\rangle$ be a bargaining problem, let $s^* \in f(S, d)$ and let $\lambda \in (0, 1)$. We must show that $s^* \in f(S, (1-\lambda)d + \lambda s^*)$. Since $s^* \in f(S, d)$, there is a negotiation problem $B = \langle u, D\rangle$ that induces $\langle S, d\rangle$ and an outcome $x^* = F(B)$ such that $s^* = u(x^*)$. Since $\lambda \in (0, 1)$, there is a time $\tau \in T$ such that $\lambda = e^{-\delta\tau}$. Since B is convex, there is an outcome $z \in X$ such that

$$\begin{aligned}
u(z) &= (1 - \lambda)u(D) + \lambda u(x^*) \\
&= (1 - \lambda)d + \lambda s^*.
\end{aligned}$$

Therefore, $\langle S, (1-\lambda)d + \lambda s^*\rangle$ is induced by $\langle (u_1, u_2), z\rangle$. Since F satisfies the tree-folding property,

$$x^* = F((u_1, u_2), z)$$

which implies that $s^* \in f(S, (1-\lambda)d + \lambda s^*)$.

Twisting: Let $\langle S, d\rangle$ be a bargaining problem and let $(s_1^*, s_2^*) \in f(S, d)$. Let $\langle S', d\rangle$ be another bargaining problem for which

$$\begin{aligned} S \setminus S' &\subseteq \{(s_1, s_2) \in \mathbb{R}^2 : s_1 > s_1^*\} \\ S' \setminus S &\subseteq \{(s_1, s_2) \in \mathbb{R}^2 : s_1 < s_1^*\}. \end{aligned}$$

Note that $s^* \in S'$. We must show that there is $(s_1', s_2') \in f(S', d)$ with $s_1' \leq s_1^*$. Since $(s_1^*, s_2^*) \in f(S, d)$, there is a negotiation problem $B = \langle (u_1, u_2), D\rangle \in \mathcal{N}$ and an outcome $x^* = F(B)$ such that S is induced by B and $(s_1^*, s_2^*) = (u_1(x^*), u_2(x^*))$. We shall define utility functions u_1' and u_2' such that $B' = \langle (u_1', u_2'), D\rangle$ induces $\langle S', d\rangle$. Let $x \in X$ such that $x \neq D$ and let $s = (s_1, s_2) = u(x)$. Consider the ray $R(x) = \{\lambda s \in \mathbb{R}^2 : \lambda \geq 0\}$ that begins at d and goes through s and define the following three numbers:

$$\begin{aligned} \lambda(x) &= \max\{\lambda \geq 0 : (1-\lambda)d + \lambda s \in S\} \\ \lambda'(x) &= \max\{\lambda \geq 0 : (1-\lambda)d + \lambda s \in S'\} \\ \alpha(x) &= \frac{\lambda'(x)}{\lambda(x)} \end{aligned}$$

Since S and S' are compact sets and since $d \in \mathrm{Int} S \cap \mathrm{Int} S'$, the above numbers are well defined for all x. Further,

$$\alpha(x) \geq 1 \Leftrightarrow S \cap R(x) \subseteq S' \cap R(x).$$

We can now define for $i = 1, 2$,

$$u_i'(x) = \begin{cases} u_i(x) & \text{if } x = D \\ \alpha(x) u_i(x) & \text{if } x \neq D. \end{cases}$$

It can be seen that with this utility functions, $B' = \langle (u_1', u_2'), D\rangle$ induces S'. Further,

1. $u_i(D) = u_i'(D)$ and $u_i(x^*) = u_i'(x^*)$, for $i = 1, 2$
2. for all $x \in X$, if x is efficient and individually rational in B, so is x is efficient in B'
3. for all efficient and individually rational points $x, y \in X$, $u_i(x) \geq u_i(y)$ if and only if $u_i'(x) \geq u_i'(y)$, for $i = 1, 2$

Let now $x \in X$ and $\tau \in T$ such that

$$(1 - e^{-\delta\tau}) u_1(D) + e^{-\delta\tau} u_1(x) = u_1(x^*) \tag{7}$$

where x^* is the outcome for which $u_i(x^*) = s_i^*$, for $i = 1, 2$. Upon reflection one realizes that $\alpha(x) \leq 1$ which implies

$$(1 - e^{-\delta\tau})u_1'(D) + e^{-\delta\tau}u_1'(x) \leq u_1'(x^*).$$

Similarly, there is τ' such that

$$(1 - e^{-\delta\tau})u_2(D) + e^{-\delta\tau'}u_2(x^*) = u_2(x). \tag{8}$$

Since $u_2(D) = u_2'(D)$, $u_2(x^*) = u_2'(x^*)$,

$$(1 - e^{-\delta\tau})u_2'(D) + e^{-\delta\tau}u_2'(x^*) \geq u_2'(x).$$

By IM of F we have $u_1'(F(B)) \geq u_1'(F(B'))$ and $u_2'(F(B)) \leq u_2'(F(B'))$. Therefore $(s_1, s_2) = u'(F(B')) \in f(S', d)$ with $s_1' \leq s_1^*$, which is exactly what we wanted to prove.
Independence of non-individually ratioinal alternatives: Let $\langle S, d\rangle$ and $\langle S', d\rangle$ be two bargaining problems such that

$$\{s \in S : s \geq d\} = \{s \in S' : s \geq d\}.$$

We want to show that $f(S, d) = f(S', d)$. Let $s^* = (s_1^*, s_2^*) \in f(S, d)$. Then, there exists an ordinal negotiation problem $B = \langle (u_1, u_2), D\rangle$ and an outcome $x^* \in X$ such that B induces $\langle S, d\rangle$ and $s_i^* = u_i(x^*)$ for $i = 1, 2$. Now build $B' = \langle (u_1', u_2'), D\rangle$ as in the proof of twisting so that B' induces $\langle S', d\rangle$. It is clear that $\mathcal{IR}(B) = \mathcal{IR}(B')$ and that u and u' coincide on $\mathcal{IR}(B)$. Consequently, by independence of non-individually rational outcomes of F we have that $x^* = F(B')$. As a result, $s^* \in f(S', d)$. $\square$

Since f si the Nash bargaining solution, we can conclude that F is the time preference negotiation solution, and the proof is complete.

4 Conclusion

In this paper, we provided a characterization of the time-preference Nash solution using, along with some standard axioms, properties that are related to preferences over outcome streams. The ordinal Nash solution introduced by Rubinstein, Safra, and Thomson (1992) and the time-preference Nash solution analyzed here are different concepts. Although a primitive of both models is an abstract set X of physical outcomes, the former solution needs preferences over lotteries on X to be defiend, while the latter needs time-preferences to be defined. Therefore the two concepts are not comparable. On the other hand, since both solutions select an outcome from the primitive set X, we can compare them when X itself consists of a set of lotteries. In this case, we've seen that both solutions may select different outcomes, unless preferences satisfy the expected utility axioms. Further, no assumptions on the within-periods risk preferences are needed for the time-preference solution to exist.

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